Exam 20/01/2022 ①

Exercice I.

1) $$X'(t) = \underbrace{\begin{pmatrix} 4 & -1 & -1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}}_{A} X(t)$$

$$C_A(\lambda) = \begin{vmatrix} (4-\lambda) & -1 & -1 \\ 1 & (2-\lambda) & -1 \\ 1 & -1 & (2-\lambda) \end{vmatrix}$$

$$= (4-\lambda)(2-\lambda)^2 + 1 + 1 + \cancel{2} \cancel{-\lambda} \cancel{-4} \cancel{+\lambda} + 2 - \lambda$$

$$= (4-\lambda)(2-\lambda)^2 + 2 - \lambda$$

$$= (2-\lambda)\left[(4-\lambda)(2-\lambda) + 1\right]$$

$$= (2-\lambda)(8 - 6\lambda + \lambda^2 + 1)$$

$$= -(\lambda-2)(\lambda^2 - 6\lambda + 9)$$

$$= -(\lambda-2)(\lambda-3)^2$$

2 val. propre simple / 3 val propre double

2) Soit $v_2 \in \mathrm{Ker}(A - 2I_3)$

$$(A - 2I_3) = \begin{pmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} \qquad \left.\begin{array}{l} 2x - y - z = 0 \\ x - z = 0 \\ x - y = 0 \end{array}\right| \longrightarrow \begin{array}{l} x = z \\ x = y \end{array}$$

On choisit $v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $E_2 = \text{Vect}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$

* Soit $v_3 \in \text{Ker}(A - 3I_3)$

$$(A - 3I_3) = \begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix}, \quad x - y - z = 0$$

Il nous faut 2 vecteurs indépendants, on prend :

$$v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad w_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$E_3 = \text{Vect}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}$$

3) On a : $P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ avec $P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

$$X'(t) = AX(t) \iff P^{-1}X'(t) = \underbrace{P^{-1}P}_{I_3}\Delta P^{-1} X(t), \quad \overbrace{P\Delta P^{-1}}^{A}$$

On pose $Y(t) = P^{-1}X(t)$, et :

$$Y'(t) = \Delta Y(t) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} Y(t)$$

Soit :

$$y_1'(t) = 2y_1(t) \longrightarrow y_1(t) = \lambda e^{2t}$$
$$y_2'(t) = 3y_2(t) \longrightarrow y_2(t) = \mu e^{3t}$$
$$y_3'(t) = 3y_3(t) \longrightarrow y_3(t) = \gamma e^{3t}$$

$$X(t) = P.Y(t) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} \lambda e^{2t} \\ \mu e^{3t} \\ \gamma e^{3t} \end{pmatrix}$$

$\lambda, \mu, \gamma \in \mathbb{R}$

Finalement :
$$\begin{cases} x_1(t) = \lambda e^{2t} + \mu e^{3t} + \gamma e^{3t} \\ x_2(t) = \lambda e^{2t} + \mu e^{3t} \\ x_3(t) = \lambda e^{2t} + \gamma e^{3t} \end{cases}$$

Exercice 2 .

$$(E) : \quad y'''(t) - 3y''(t) + 3y'(t) - y(t) = t^2 e^t$$

$$y(0) = 1 \;,\; y'(0) = 0 \;,\; y''(0) = -2$$

On pose $Y(s) = \mathcal{L}\{y(t)\}(s)$

1) ⊛ $\mathcal{L}\{y'''(t)\}(s) = s^3 Y(s) - s^2 y(0) - s y'(0) - y''(0)$
$$= s^3 Y(s) - s^2 - 2$$

⊛ $\mathcal{L}\{y''(t)\}(s) = s^2 Y(s) - s y(0) - y'(0)$
$$= s^2 Y(s) - s$$

⊛ $\mathcal{L}\{y'(t)\}(s) = sY(s) - y(0) = sY(s) - 1$

⊛ $\mathcal{L}\{t^2 e^t\}(s) = \mathcal{L}\{t^2\}(s-1) = \dfrac{2}{(s-1)^3}$

En transformant (E) :

$$s^3 Y(s) - s^2 + 2 - 3(s^2 Y(s) - s) + 3(sY(s) - 1) - Y(s) = \frac{2}{(s-1)^3}$$

(4)

$$Y(s)\Big[\underbrace{s^3-3s^2+3s-1}_{(s-1)^3}\Big]\ \underbrace{-s^2+2+3s-3}_{-s^2+3s-1} = \frac{2}{(s-1)^3}$$

$$\begin{cases} f(s) = (s-1)^3 \\ g(s) = -s^2+3s-1 \qquad , \quad f(s)Y(s)+g(s) = h(s) \\ h(s) = \dfrac{2}{(s-1)^3} \end{cases}$$

2) $$Y(s) = \frac{1}{(s-1)^3}\left(\frac{2}{(s-1)^3} + s^2-3s+1\right)$$

$$= \frac{2}{(s-1)^6} + \frac{1}{(s-1)^3}\left(\overbrace{(s-1)^2+2s\cancel{-1}}-3s\cancel{+1}\right)$$

$$= \frac{2}{(s-1)^6} + \frac{(s-1)^2-s}{(s-1)^3}$$

$$= \frac{2}{(s-1)^6} + \frac{1}{(s-1)} - \underbrace{\frac{s}{(s-1)^3}}_{-\frac{(s-1)+1}{(s-1)^3}}$$

$$Y(s) = \frac{2}{(s-1)^6} + \frac{1}{(s-1)} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3}$$

$$= \frac{2}{5!}\cdot\frac{5!}{(s-1)^6} + \frac{1}{(s-1)} - \frac{1}{(s-1)^2} - \frac{1}{2}\cdot\frac{2}{(s-1)^3}$$

Par transformation inverse :

$$y(t) = e^t . \mathcal{L}^{-1}\left\{ \frac{2}{5!} \cdot \frac{5!}{s^6} + \frac{1}{s} - \frac{1}{s^2} - \frac{1}{2} \cdot \frac{2}{s^3} \right\}(t)$$

$$= e^t \left(\frac{2}{5!} t^5 + 1 - t - \frac{t^2}{2} \right)$$

$$\boxed{y(t) = e^t \left(\frac{1}{60} t^5 - \frac{1}{2} t^2 - t + 1 \right)}$$

Exercice 3

1) $f(x) = e^{-|x|}$

$$\mathcal{F}\{f(x)\}(\nu) = \int_{-\infty}^{+\infty} f(x) e^{-2i\pi\nu x} dx = \int_{-\infty}^{+\infty} e^{-|x|} e^{-2i\pi\nu x} dx$$

$$= \int_{-\infty}^{0} e^{x} e^{-2i\pi\nu x} dx + \int_{0}^{+\infty} e^{-x} e^{-2i\pi\nu x} dx$$

$$= \int_{-\infty}^{0} e^{(1-2i\pi\nu)x} dx + \int_{0}^{+\infty} e^{-(1+2i\pi\nu)x} dx$$

$$= \left[\frac{e^{(1-2i\pi\nu)x}}{1-2i\pi\nu} \right]_{-\infty}^{0} + \left[-\frac{e^{(1+2i\pi\nu)x}}{1+2i\pi\nu} \right]_{0}^{+\infty}$$

$$\mathcal{F}[f(x)](\lambda) = \frac{1}{1-2i\pi\lambda} - \left(-\frac{1}{1+2i\pi\lambda}\right)$$

$$= \frac{1}{1-2i\pi\lambda} + \frac{1}{1+2i\pi\lambda} = \frac{\cancel{1+2i\pi\lambda} + \cancel{1-2i\pi\lambda}}{(1-2i\pi\lambda)(1+2i\pi\lambda)}$$

$$\boxed{\mathcal{F}[f(x)](\lambda) = \frac{2}{1+4\pi^2\lambda^2}}$$

2) $g(x) = |x|e^{-|x|}$

$$\mathcal{F}[g(x)](\lambda) = \int_{-\infty}^{0} -x e^{(1-2i\pi\lambda)x}\,dx + \int_{0}^{+\infty} x e^{-(1+2i\pi\lambda)x}\,dx$$

<u>I.P.P.</u> $u = -x$, $u' = -1$ $\qquad$ $u = x$, $u' = 1$

$v' = e^{(1-2i\pi\lambda)x}$, $v = \frac{e^{(1-2i\pi\lambda)x}}{1-2i\pi\lambda}$ $\qquad$ $v' = e^{-(1+2i\pi\lambda)x}$, $v = -\frac{e^{-(1+2i\pi\lambda)x}}{1+2i\pi\lambda}$

$$\mathcal{F}[g(x)](\lambda) = \underbrace{\cancel{\left[\frac{-xe^{(1-2i\pi\lambda)x}}{1-2i\pi\lambda}\right]_{-\infty}^{0}}}_{=0+0} + \int_{-\infty}^{0} \frac{e^{(1-2i\pi\lambda)x}}{1-2i\pi\lambda}\,dx$$

$$+ \underbrace{\cancel{\left[-\frac{xe^{-(1+2i\pi\lambda)x}}{1+2i\pi\lambda}\right]_{0}^{+\infty}}}_{=0+0} + \int_{0}^{+\infty} \frac{e^{-(1+2i\pi\lambda)x}}{1+2i\pi\lambda}\,dx$$

$$= \left[\frac{e^{(1-2i\pi\lambda)x}}{(1-2i\pi\lambda)^2}\right]_{-\infty}^{0} + \left[-\frac{e^{-(1+2i\pi\lambda)x}}{(1+2i\pi\lambda)^2}\right]_{0}^{+\infty}$$

$$\mathcal{F}\{g(x)\}(\lambda) = \frac{1}{(1-2i\pi\lambda)^2} + \frac{1}{(1+2i\pi\lambda)^2}$$

$$= \frac{1 + \cancel{4i\pi\lambda} - 4\pi^2\lambda^2 + 1 - \cancel{4i\pi\lambda} - 4\pi^2\lambda^2}{(1-2i\pi\lambda)^2(1+2i\pi\lambda)^2}$$

$$\boxed{\mathcal{F}\{g(x)\}(\lambda) = \frac{2 - 8\pi^2\lambda^2}{(1+4\pi^2\lambda^2)^2}}$$

(3) $h(x) = (1-|x|)\, e^{-|x|} = f(x) - g(x)$

$$\mathcal{F}\{h(x)\}(\lambda) = \mathcal{F}\{f(x)\}(\lambda) - \mathcal{F}\{g(x)\}(\lambda)$$

$$= \frac{2}{1+4\pi^2\lambda^2} - \frac{2-8\pi^2\lambda^2}{(1+4\pi^2\lambda^2)^2}$$

$$= \frac{2 + 8\pi^2\lambda^2 - 2 + 8\pi^2\lambda^2}{(1+4\pi^2\lambda^2)^2}$$

$$= \frac{16\pi^2\lambda^2}{(1+4\pi^2\lambda^2)^2}$$

$$\boxed{\mathcal{F}\{h(x)\}(\lambda) = 4\cdot\frac{(2\pi\lambda)^2}{(1+(2\pi\lambda)^2)^2}}$$

(4) $$\mathcal{F}\left\{\frac{x^2}{(1+x^2)^2}\right\}(\lambda) = \mathcal{F}\left\{\frac{1}{4}\cdot\mathcal{F}\{h\}\left(\frac{x}{2\pi}\right)\right\}(\lambda)$$

$$= \frac{1}{4}\cdot 2\pi\,\mathcal{F}\left\{\mathcal{F}\{h\}(x)\right\}(2\pi\lambda)$$

(Changt d'échelle)

$$= \frac{\pi}{2}\cdot h(-2\pi\lambda) \quad \text{(Transfo de la transfo)}$$

$$= \frac{\pi}{2}\cdot\left(1-|2\pi\lambda|\right)e^{-|2\pi\lambda|}$$

$$\boxed{\mathcal{F}\left\{\frac{x^2}{(1+x^2)^2}\right\}(\lambda) = \frac{\pi}{2}\left(1-2\pi|\lambda|\right)e^{-2\pi|\lambda|}}$$

(5) $\hat{u}(\lambda,y) = \mathcal{F}\{u(x,y)\}(\lambda)$

En transformant l'E.D.P :

$$(2i\pi\lambda)^2\,\hat{u}(\lambda,y) + \frac{\partial^2}{\partial y^2}\hat{u}(\lambda,y) = 0$$

Soit $\dfrac{\partial^2\hat{u}}{\partial y^2} - 4\pi^2\lambda^2\,\hat{u} = 0$

Donc : $\underline{\hat{u}(\lambda,y) = A e^{2\pi|\lambda|y} + B e^{-2\pi|\lambda|y}}$

$u(x,y) \xrightarrow[y\to+\infty]{} 0 \implies \hat{u}(\lambda,y) \xrightarrow[y\to+\infty]{} 0 \implies \underline{A=0}$

Finalement : $\hat{u}(\lambda,y) = B\, e^{-2\pi|\lambda| y}$

$$= \hat{u}(\lambda,0)\, e^{-2\pi|\lambda| y}$$

$$\hat{u}(\lambda,0) = \mathcal{F}\{u(x,0)\}(\lambda)$$

$$= \mathcal{F}\left\{\frac{x^2}{(1+x^2)^2}\right\}(\lambda) = \frac{\pi}{2}\left(1-2\pi|\lambda|\right)e^{-2\pi|\lambda|}$$

$$\hat{u}(\lambda,y) = \frac{\pi}{2}\left(1-2\pi|\lambda|\right)e^{-2\pi(1+y)|\lambda|}$$

Il nous faut revenir à $u(x,y)$, on va développer et utiliser la transformée de la transformée ...

$$\hat{u}(\nu,y) = \frac{\pi}{2} e^{-|2\pi(1+y)\nu|} - \frac{\pi}{2} \cdot 2\pi|\nu| e^{-|2\pi(1+y)\nu|}$$

$$= \frac{\pi}{2} f(2\pi(1+y)\nu) - \frac{\pi}{2(1+y)} \cdot |2\pi(1+y)\nu| e^{-|2\pi(1+y)\nu|}$$

$$= \underbrace{\frac{\pi}{2} f(2\pi(1+y)\nu)}_{\hat{u}_1(\nu,y)} - \underbrace{\frac{\pi}{2(1+y)} g(2\pi(1+y)\nu)}_{\hat{u}_2(\nu,y)}$$

$$\circledast \; u_1(x,y) = \mathcal{F}\{\hat{u}_1(\nu,y)\}(-x)$$

$$= \frac{\pi}{2} \mathcal{F}\{f(2\pi(1+y)\nu)\}(-x)$$

$$= \frac{\pi}{2} \cdot \frac{1}{2\pi(1+y)} \mathcal{F}\{f\}\left(-\frac{x}{2\pi(1+y)}\right)$$

$$= \frac{1}{4(1+y)} \cdot \frac{2}{1 + 4\pi^2 \frac{x^2}{4\pi^2(1+y)^2}}$$

$$u_1(x,y) = \frac{1}{2} \cdot \frac{1+y}{(1+y)^2 + x^2}$$

$$u_2(x,y) = \mathcal{F}\left\{ \hat{u}_2(\xi,y) \right\}(-x)$$

$$= -\frac{\pi}{2(1+y)} \mathcal{F}\left\{ g\left(2\pi(1+y)\xi\right) \right\}(-x)$$

$$= -\frac{\cancel{\pi}}{2(1+y)} \cdot \frac{1}{2\cancel{\pi}(1+y)} \mathcal{F}\left\{ g \right\}\left(-\frac{x}{2\pi(1+y)} \right)$$

$$= -\frac{1}{4(1+y)^2} \cdot \frac{2 - 8\pi^2\left(\frac{x}{2\pi(1+y)}\right)^2}{\left(1 + 4\pi^2\left(\frac{x}{2\pi(1+y)}\right)^2\right)^2}$$

$$= \frac{2 - \frac{\cancel{8}^{\,2}\,\cancel{\pi^2}\,x^2}{\cancel{4}\,\cancel{\pi^2}(1+y)^2}}{\left(1 + \cancel{4\pi^2}\frac{x^2}{\cancel{4\pi^2}(1+y)^2}\right)^2} \cdot \frac{(-1)}{4(1+y)^2}$$

$$= \frac{1 - \frac{x^2}{(1+y)^2}}{\left(1 + \frac{x^2}{(1+y)^2}\right)^2} \quad \frac{+\cancel{2}\,(-1)}{{}_2\cancel{4}(1+y)^2}$$

$$= \frac{(1+y)^2 - x^2}{2\left((1+y)^2 + x^2\right)^2} \cdot (-1)$$

$$u_2(x,y) = \frac{1}{2} \cdot \frac{x^2 - (1+y)^2}{\left(x^2 + (1+y)^2\right)^2}$$

<u>Finalement</u>: $u(x,y) = u_1(x,y) + u_2(x,y)$

$$u(x,y) = \frac{1}{2}\left(\frac{1+y}{x^2+(1+y)^2} + \frac{x^2-(1+y)^2}{\left(x^2+(1+y)^2\right)^2}\right)$$

$$= \frac{1}{2}\;\frac{x^2(1+y)+(1+y)^3+x^2-(1+y)^2}{\left(x^2+(1+y)^2\right)^2}$$

$$= \frac{1}{2}\cdot\frac{x^2(2+y)+(1+y)^2(\cancel{1}+y\cancel{-1})}{\left(x^2+(1+y)^2\right)^2}$$

$$\boxed{u(x,y) = \frac{1}{2}\cdot\frac{x^2(2+y)+y(1+y)^2}{\left(x^2+(1+y)^2\right)^2}}$$